

COMMON HYPERCYCLIC FUNCTIONS FOR TRANSLATION OPERATORS WITH LARGE GAPS

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ABSTRACT. Let $H(\mathbb{C})$ be the set of entire functions endowed with the topology of local uniform convergence. Fix a sequence of non-zero complex numbers (λ_n) , $|\lambda_n| \rightarrow +\infty$, which satisfies the following property: for every $M > 0$ there exists a subsequence (μ_n) of (λ_n) such that

- (i) $|\mu_{n+1}| - |\mu_n| > M$ for every $n = 1, 2, \dots$ and
- (ii) $\sum_{n=1}^{+\infty} \frac{1}{|\mu_n|} = +\infty$.

We prove that there exists a residual set $G \subset H(\mathbb{C})$ such that for every $f \in G$ and every non-zero complex number a the set $\{f(z + \lambda_n a) : n = 1, 2, \dots\}$ is dense in $H(\mathbb{C})$. This answers in the affirmative Question 1 in [28] and it also provides an extension of a theorem due to Costakis and Sambarino in [31].

1. INTRODUCTION

We start by fixing some standard notation and terminology. The symbols $\mathbb{N} = \{1, 2, \dots\}$, \mathbb{Q} , \mathbb{R} , \mathbb{C} stand for the sets of natural, rational, real and complex numbers respectively. By $H(\mathbb{C})$ we denote the set of entire functions endowed with the topology of local uniform convergence. For a subset A of $H(\mathbb{C})$, \overline{A} denotes the closure of A with respect to the topology of local uniform convergence. Let X be a topological vector space. A subset G of a X is called G_δ if it can be written as a countable intersection of open sets in X and a subset Y of X is called residual if it contains a G_δ and dense subset of X . The symbol ∞ whenever appears in the present work denotes the complex infinity.

Let $(T_n : X \rightarrow X)$ be a sequence of continuous linear operators on a topological vector space X . If $(T_n(x))_{n \geq 1}$ is dense in X for some $x \in X$, then x is called *hypercyclic* for (T_n) and we say that (T_n) is *hypercyclic* [12], [40]. The symbol

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$HC(\{T_n\})$ stands for the collection of all hypercyclic vectors for (T_n) . In the case where the sequence (T_n) comes from the iterates of a single operator $T : X \rightarrow X$, i.e. $T_n := T^n$, then we simply say that T is *hypercyclic* and x is *hypercyclic* for T . If $T : X \rightarrow X$ is hypercyclic then the symbol $HC(T)$ stands for the collection of all hypercyclic vectors for T . A simple consequence of Baire's category theorem is that for every continuous linear operator T on a separable topological vector space X , if $HC(T)$ is non-empty then it is necessarily $(G_\delta$ and) dense. For an account of results on the subject of hypercyclicity we refer to the recent books [12], [40], see also the very influential survey article [38].

In the present work we deal with translation operators. For every $a \in \mathbb{C} \setminus \{0\}$ consider the translation operator $T_a : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ defined by

$$T_a(f)(z) = f(z + a), \quad f \in H(\mathbb{C}).$$

An old result of Birkhoff [19] says that there exist entire functions the integer translates of which are dense in the space of all entire functions endowed with the topology of local uniform convergence. In other words T_1 is hypercyclic. Actually, it is not difficult to see that for every $a \in \mathbb{C} \setminus \{0\}$, T_a is hypercyclic and hence $HC(T_a)$ is G_δ and dense in $H(\mathbb{C})$. Costakis and Sambarino [31] strengthened Birkhoff's result by showing that the family $\{T_a \mid a \in \mathbb{C} \setminus \{0\}\}$ has a residual set of common hypercyclic vectors i.e., the set $\bigcap_{a \in \mathbb{C} \setminus \{0\}} HC(\{T_{na}\})$ is residual in $H(\mathbb{C})$. In particular, it is non-empty. Of course, what makes their result non-trivial is the uncountable range of a . At this point, let us mention a relevant observation due to Bayart and Matheron, [12], [13]: suppose X is a Fréchet space and $\{S_{a,n} \mid a \in A, n \in \mathbb{N}\}$ is a collection of sequences of continuous linear operators on X , labelled by the elements a of a set A . If A is a σ -compact topological space, the maps $a \rightarrow S_{a,n}$ are *SOT*-continuous and each sequence $(S_{a,n})_{n \in \mathbb{N}}$ has a dense set of hypercyclic vectors then either $\bigcap_{a \in A} HC(\{S_{a,n}\}) = \emptyset$ or $\bigcap_{a \in A} HC(\{S_{a,n}\})$ is a dense G_δ -set in X . This observation applies to all the collections of operators considered in our work.

Let us now come to the main subject of our paper. Recall that the set $\bigcap_{a \in \mathbb{C} \setminus \{0\}} HC(\{T_{na}\})$ is residual in $H(\mathbb{C})$, [31]. Subsequently, Costakis [28] asked whether, in this result, the sequence (n) can be replaced by more general sequences (λ_n) of non-zero complex numbers. In this direction Costakis [28] showed that, if the sequence (λ_n) satisfies the following condition (Σ) : for every $M > 0$ there exists a subsequence (μ_n) of (λ_n) such that

- (i) $|\mu_{n+1}| - |\mu_n| > M$ for every $n = 1, 2, \dots$ and

$$(ii) \sum_{n=1}^{+\infty} \frac{1}{|\mu_n|} = +\infty ,$$

then the desired conclusion holds if we restrict attention to $a \in C(0, 1) := \{z \in \mathbb{C} / |z| = 1\}$, that is the set $\bigcap_{a \in C(0,1)} HC(\{T_{\lambda_n a}\})$ is residual in $H(\mathbb{C})$. In view of the above, Costakis led to the following question, see Question 1 in [28].

Question 1.1. *Let (λ_n) be a sequence of non-zero complex numbers tending to infinity which also satisfies condition (Σ) . Is it true that the set $\bigcap_{a \in \mathbb{C} \setminus \{0\}} HC(\{T_{\lambda_n a}\})$ is residual in $H(\mathbb{C})$, hence non-empty?*

Our main task is to give an affirmative answer to Question 1.1 by proving the following

Theorem 1.1. *Fix a sequence of non-zero complex numbers $\Lambda = (\lambda_n)$ that tends to infinity and satisfies the above condition (Σ) . Then $\bigcap_{a \in \mathbb{C} \setminus \{0\}} HC(\{T_{\lambda_n a}\})$ is a G_δ and dense subset of $H(\mathbb{C})$.*

It is worth to mention here that one is forced to impose certain natural restrictions on the sequence (λ_n) in order to conclude that the set $\bigcap_{a \in \mathbb{C} \setminus \{0\}} HC(\{T_{\lambda_n a}\})$ is non-empty. Indeed, in [32] the authors show that if $\liminf_n \frac{|\lambda_{n+1}|}{|\lambda_n|} > 2$ then $\bigcap_{a \in \mathbb{C} \setminus \{0\}} HC(\{T_{\lambda_n a}\}) = \emptyset$. In particular, $\bigcap_{a \in \mathbb{C} \setminus \{0\}} HC(\{T_{e^n a}\}) = \emptyset$. However, for sequences (λ_n) with

$$1 < \liminf_n \frac{|\lambda_{n+1}|}{|\lambda_n|} \leq 2$$

it is not known whether

$$\bigcap_{a \in \mathbb{C} \setminus \{0\}} HC(\{T_{\lambda_n a}\}) = \emptyset,$$

although it is plausible to conjecture that this is the case. In particular, we do not know what happens when $\lambda_n = 2^n$ or $\lambda_n = (3/2)^n$. This work can be seen as a try to understand the nature of this restriction. In any case, it seems a quite difficult problem to fully characterize the sequences (λ_n) for which the conclusion of Theorem 1.1 holds.

We stress that Theorem 1.1 complements the main result from our recent work in [48]. In [48] we showed that the conclusion of Theorem 1.1 holds for sequences (λ_n) satisfying another type of condition different from (Σ) ; this condition, which we call it (Σ') , is also not very restrictive, in the sense that it still allows sequences (λ_n) with “large gaps”. To avoid extra notation and to keep the introduction in a compact form, we postpone the definition of condition (Σ') till section 6. We note

that although sequences of polynomial type of degree bigger than one, such as (n^2) , (n^3) , $(n+n^3)$, (n^4+n^5) and so on, clearly do not satisfy condition (Σ) they do satisfy (Σ') . On the other hand there exist sequences satisfying (Σ') which do not satisfy (Σ) . However, there exist sequences satisfying both conditions (Σ) and (Σ') . All these are explained in full detail in Section 6.

A few words about the proof of Theorem 1.1. Of course the main argument uses Baire's category theorem, but in order to do so the first and most difficult thing is to construct a suitable two dimensional partition on a given sector of the plane. After, to each point of the partition we assign a suitable closed disk of constant radius so that these disks are pairwise disjoint and their union almost fills the sector. Having done these steps we are ready for the final argument which involves a standard use of Runge's or Mergelyan's approximation theorem along with Baire's theorem. It is important to say that in our framework one cannot use Ansari's theorem [4], as Costakis and Sambarino did in their proof, since now the sequence (λ_n) lacks the semigroup structure, i.e. $\lambda_n + \lambda_m \neq \lambda_{n+m}$ in general. Actually, this was the reason that led us to seek higher order partitions in order to make things work. Overall, we elaborate on the work of Costakis and Sambarino and we offer a general strategy how to construct two dimensional partitions relevant to our problem. In general, our proof shares certain similarities with the proof of the main result in [48] and so we feel that the interested reader will get a more clear and integrated picture by reading in parallel the present paper and paper [48]. However, the methods of constructing the partitions in the present paper and [48] differentiate drastically. The reason for this, is that always the partition reflects the structure of the sequence (λ_n) . The construction of the partition in [48] is very tight and quite delicate and comes from our effort to deal firstly with the most natural sequence which fails condition (Σ) , namely the sequence (n^2) . It is also evident that there is a huge distance between sequences satisfying condition (Σ) and the sequences satisfying condition (Σ') , see section 6. Of course, it would be desirable to exhibit a condition and a corresponding partition, if any, which imply the main result of the present paper as well as the main result in [48]. Unfortunately, this is unclear to us.

There are several recent results concerning either the existence or the non-existence of common hypercyclic vectors for uncountable families of operators, such as weighted shifts, adjoints of multiplication operators, differentiation and composition operators; see for instance, [1], [6]-[14], [16] [21]-[25], [27]-[32], [34], [40], [41], [42], [44], [46], [47], [48].

Our paper is organized as follows. The proof of Theorem 1.1 has several steps and occupies Sections 2-5. Finally, in Section 6 we compare Theorem 1.1 with the main result from [48] and we exhibit examples of sequences which illustrate our main theorem.

2. A REDUCTION OF THEOREM 1.1

Let us now describe the steps for the proof of Theorem 1.1. Consider the sectors

$$S_n^k := \left\{ a \in \mathbb{C} \mid \exists r \in \left[\frac{1}{n}, n \right] \text{ and } t \in \left[\frac{k}{4}, \frac{k+1}{4} \right] \text{ such that } a = re^{2\pi it} \right\}$$

for $k = 0, 1, 2, 3$ and $n = 2, 3, \dots$. Since

$$\bigcap_{a \in \mathbb{C} \setminus \{0\}} HC(\{T_{\lambda_n a}\}) = \bigcap_{k=0}^3 \bigcap_{n=2}^{+\infty} \bigcap_{a \in S_n^k} HC(\{T_{\lambda_n a}\}),$$

an appeal of Baire's category theorem reduces Theorem 1.1 to the following.

Proposition 2.1. *Fix a sequence (λ_n) of non-zero complex numbers that tends to infinity which satisfies the above condition (Σ) . Fix four real numbers $r_0, R_0, \theta_0, \theta_T$ such that $0 < r_0 < 1 < R_0 < +\infty$, $0 \leq \theta_0 < \theta_T \leq 1$, $\theta_T - \theta_0 = \frac{1}{4}$ and consider the sector S defined by*

$$S := \{a \in \mathbb{C} \mid \text{there exist } r \in [r_0, R_0] \text{ and } t \in [\theta_0, \theta_T] \text{ such that } a = re^{2\pi it}\}.$$

Then $\bigcap_{a \in S} HC(\{T_{\lambda_n a}\})$ is a G_δ and dense subset of $H(\mathbb{C})$.

For the proof of Proposition 2.1 we introduce some notation which will be carried out throughout this paper. Let (p_j) , $j = 1, 2, \dots$ be a dense sequence of $H(\mathbb{C})$, (for instance, all the polynomials in one complex variable with coefficients in $\mathbb{Q} + i\mathbb{Q}$). For every $m, j, s, k \in \mathbb{N}$ we consider the set

$$E(m, j, s, k) := \left\{ f \in H(\mathbb{C}) \mid \forall a \in S \ \exists n \in \mathbb{N}, n \leq m : \sup_{|z| \leq k} |f(z + \lambda_n a) - p_j(z)| < \frac{1}{s} \right\}.$$

By Baire's category theorem and the three lemmas stated below, Proposition 2.1 readily follows.

Lemma 2.1.

$$\bigcap_{a \in S} HC(\{T_{\lambda_n a}\}) = \bigcap_{j=1}^{+\infty} \bigcap_{s=1}^{+\infty} \bigcap_{k=1}^{+\infty} \bigcup_{m=1}^{+\infty} E(m, j, s, k).$$

Lemma 2.2. *For every $m, j, s, k \in \mathbb{N}$ the set $E(m, j, s, k)$ is open in $H(\mathbb{C})$.*

Lemma 2.3. *For every $j, s, k \in \mathbb{N}$ the set $\bigcup_{m=1}^{+\infty} E(m, j, s, k)$ is dense in $H(\mathbb{C})$.*

The proof of Lemma 2.1 is in [48]. The proof of Lemma 2.2 is similar to that in Lemma 9 of [31] and it is omitted.

We now move on to Lemma 2.3. This lemma is the heart of our argument and its proof occupies the next three sections.

3. CONSTRUCTION OF THE PARTITION OF THE SECTOR \mathcal{S}

For the sequel we fix four positive numbers c_1, c_2, c_3, c_4 such that $c_1 > 1$, $c_2 \in (0, 1)$, $c_3 > 1$, $c_4 > 1$, where $c_3 := \frac{c_4}{r_0 c_2}$, $c_1 := 4(c_3 + 1)$. We also consider four positive real numbers $\theta_0, \theta_T, r_0, R_0$ as in Proposition 2.1 and a sequence $\Lambda = (\lambda_n)$ of non zero complex numbers which satisfies condition (Σ) and such that $\lambda_n \rightarrow \infty$ as $n \rightarrow +\infty$. After the definition of the above numbers we fix a subsequence (μ_n) of (λ_n) such that:

$$|\mu_n| > c_1, |\mu_{n+1}| - |\mu_n| > c_1 \text{ for every } n = 1, 2, \dots \text{ and } \sum_{k=1}^{+\infty} \frac{1}{|\mu_k|} = +\infty.$$

3.1. Step 1. Partitions of the interval $[\theta_0, \theta_T]$.

In this step we succeed the elementary structure of our construction. The following two steps are based in this first one. For every positive integer m we shall construct a corresponding partition Δ_m of $[\theta_0, \theta_T]$. So, let $m \in \mathbb{N}$ be fixed.

The condition $\sum_{n=1}^{+\infty} \frac{1}{|\mu_n|} = +\infty$ implies that for every positive integer $m = 1, 2, \dots$ there exists the minimum natural number $m_1(m)$ such that:

$$(3.1) \quad \sum_{k=m}^{m_1(m)} \frac{1}{|\mu_k|} > c_3 \cdot \frac{1}{|\mu_m|}.$$

Clearly $m_1(m) \geq m + 1$ for every $m = 1, 2, \dots$ because $c_3 > 1$. We define the numbers $\theta_0^{(m)} := \theta_0$, $\theta_1^{(m)} := \theta_0^{(m)} + \frac{c_2}{|\mu_m|}$, $\theta_2^{(m)} := \theta_1^{(m)} + \frac{c_2}{|\mu_{m+1}|}$, \dots , $\theta_{m_1(m)-m+1}^{(m)} := \theta_{m_1(m)-m}^{(m)} + \frac{c_2}{|\mu_{m_1(m)}|}$, or generally:

$$(3.2) \quad \theta_{n+1}^{(m)} := \theta_n^{(m)} + \frac{c_2}{|\mu_{m+n}|}, \quad n = 0, 1, \dots, m_1(m) - m,$$

where $m_1(m) - m \geq 1$. Define

$$\sigma_m := \theta_{m_1(m)-m+1}^{(m)} - \theta_0.$$

Now let any positive integer ν with

$$\nu > m_1(m) - m + 1.$$

For such a ν there exists a unique pair $(k, j) \in \mathbb{N}^2$, where $j \in \{0, 1, \dots, m_1(m) - m\}$, such that:

$$\nu = k(m_1(m) - m + 1) + j.$$

We define

$$\theta_\nu^{(m)} := \theta_j^{(m)} + k\sigma_m.$$

It is obvious that $\lim_{\nu \rightarrow +\infty} \theta_\nu^{(m)} = +\infty$ and the sequence $(\theta_\nu^{(m)})_\nu$ is strictly increasing, in respect to ν . So there exists a maximum natural number $\nu_m \in \mathbb{N}$ such that $\theta_{\nu_m}^{(m)} \leq \theta_T$. We set

$$\Delta_m := \{\theta_0^{(m)}, \theta_1^{(m)}, \dots, \theta_{\nu_m}^{(m)}\}.$$

It holds that $\nu_m \geq m_1(m) - m + 1$ (see Lemma 3.1).

3.2. Step 2. Partitions of the arc $\phi_r([\theta_0, \theta_T])$.

Consider the function $\phi : [\theta_0, \theta_T] \times (0, +\infty) \rightarrow \mathbb{C}$ given by

$$\phi(t, r) := re^{2\pi it}, \quad (t, r) \in [\theta_0, \theta_T] \times (0, +\infty)$$

and for every $r > 0$ we define the corresponding curve $\phi_r : [\theta_0, \theta_T] \rightarrow \mathbb{C}$ by

$$\phi_r(t) := \phi(t, r), \quad t \in [\theta_0, \theta_T].$$

For any given positive integer m , $\phi_r(\Delta_m)$ is a partition of the arc $\phi_r([\theta_0, \theta_T])$, where Δ_m is the partition of the interval $[\theta_0, \theta_T]$ constructed in Step 1. For every $r > 0$, $m \in \mathbb{N}$ define

$$\mathcal{P}_0^{r,m} := \phi_r(\Delta_m)$$

which we call partition of the arc $\phi_r([\theta_0, \theta_T])$ with height r , density m and order 0.

3.3. Step 3. The final partition.

Consider the partition $\mathcal{P}_0^{r_0,1}$ from the previous step, Step 2 and set

$$(3.3) \quad r_1 := r_0 + \frac{c_2}{|\mu_{m_1(1)}|}.$$

After, we consider the partition $\mathcal{P}_0^{r_1, m_1(1)+1}$ and we set

$$m_2 := m_1(m_1(1) + 1),$$

$$r_2 := r_1 + \frac{c_2}{|\mu_{m_2}|}.$$

Inductively we define two sequences (r_ν) , $\nu = 0, 1, 2, \dots$, (m_ν) , $\nu = 2, \dots$, as follows: r_0, r_1, r_2 and m_2 are as above, see (3.3). Suppose that we have constructed

the numbers m_ν , r_ν for some $\nu \geq 2$. Then, taking into account the partition $\mathcal{P}_0^{r_\nu, m_\nu+1}$, we set

$$(3.4) \quad m_{\nu+1} = m_1(m_\nu + 1)$$

and

$$(3.5) \quad r_{\nu+1} := r_\nu + \frac{c_2}{|\mu_{m_{\nu+1}}|}.$$

For the next step, consider the partition $\mathcal{P}_0^{r_{\nu+1}, m_{\nu+1}+1}$. We will prove in the next subsection that $\lim_{\nu \rightarrow +\infty} r_\nu = +\infty$. Therefore there exists a maximum natural number $\nu_0 \in \mathbb{N}$ such that $r_{\nu_0} \leq R_0$ because the sequence (r_ν) is strictly increasing. In view of the above, we define

$$\mathcal{P} := \mathcal{P}_0^{r_0, 1} \cup \left(\bigcup_{\nu=1}^{\nu_0} \mathcal{P}_0^{r_\nu, m_\nu+1} \right),$$

which is the desired partition of our sector S .

3.4. Properties of the partitions.

Lemma 3.1. *Let some fixed $m \in \mathbb{N}$. Then*

$$\sigma_m = \theta_{m_1(m)-m+1}^{(m)} - \theta_0 < \frac{1}{4}.$$

In particular, $\nu_m \geq m_1(m) - m + 1$.

Proof. By the definition of the numbers θ_j^m , $j = 0, 1, \dots, m_1(m) - m + 1$ we have

$$(3.6) \quad \theta_{m_1(m)-m+1}^{(m)} - \theta_0 = c_2 \cdot \sum_{k=m}^{m_1(m)} \frac{1}{|\mu_k|},$$

and by the definition of the number $m_1(m)$ it follows that

$$(3.7) \quad \sum_{k=m}^{m_1(m)} \frac{1}{|\mu_k|} \leq c_3 \cdot \frac{1}{|\mu_m|} + \frac{1}{|\mu_{m_1(m)}|} < (c_3 + 1) \frac{1}{|\mu_m|}.$$

Our hypotheses imply $c_1 = 4(c_3 + 1)$ and $|\mu_m| > c_1 = 4(c_3 + 1) > 4c_2(c_3 + 1)$, because $c_2 \in (0, 1)$. This gives

$$(3.8) \quad \frac{c_3 + 1}{|\mu_m|} < \frac{1}{4c_2}.$$

Thus, (3.6), (3.7) and (3.8) yield $\sigma_m < \frac{1}{4}$ and the proof is complete. \blacksquare

Lemma 3.2. $\lim_{\nu \rightarrow +\infty} r_\nu = +\infty$.

Proof. Below, let us rewrite the relations that define the numbers (r_ν) , $\nu = 0, 1, 2, \dots$

$$(3.9) \quad r_1 = r_0 + \frac{c_2}{|\mu_{m_1(1)}|},$$

$$(3.10) \quad r_2 = r_1 + \frac{c_2}{|\mu_{m_2}|},$$

$$(3.11) \quad r_{\nu+1} = r_\nu + \frac{c_2}{|\mu_{m_{\nu+1}}|}, \quad \nu = 1, 2, \dots,$$

where $m_2 := m_1(m_1(1) + 1)$, see subsection 3.3. Equalities (3.9), (3.10), (3.11) imply

$$(3.12) \quad r_\nu = r_0 + c_2 \cdot \sum_{k=1}^{\nu} \frac{1}{|\mu_{m_k}|} \quad \text{for } \nu = 1, 2, \dots, \quad \text{where } m_1 = m_1(1).$$

By the definitions of $m_1(1)$, m_2 we have

$$(3.13) \quad \sum_{k=1}^{m_1(1)} \frac{1}{|\mu_k|} \leq c_3 \cdot \frac{1}{|\mu_1|} + \frac{1}{|\mu_{m_1(1)}|} < (c_3 + 1) \frac{1}{|\mu_1|},$$

$$(3.14) \quad \sum_{k=m_1(1)+1}^{m_2} \frac{1}{|\mu_k|} \leq c_3 \cdot \frac{1}{|\mu_{m_1(1)+1}|} + \frac{1}{|\mu_{m_2}|} < (c_3 + 1) \cdot \frac{1}{|\mu_{m_1(1)}|}.$$

Inductively, for every $\nu \geq 2$ we get

$$(3.15) \quad \sum_{k=m_{\nu}+1}^{m_{\nu+1}} \frac{1}{|\mu_k|} \leq c_3 \cdot \frac{1}{|\mu_{m_{\nu}+1}|} + \frac{1}{|\mu_{m_{\nu+1}}|} < (c_3 + 1) \frac{1}{|\mu_{m_{\nu}}|}$$

because the sequence $(|\mu_n|)$ is strictly increasing. So by (3.13), (3.14) and (3.15) we conclude that

$$(3.16) \quad \sum_{k=1}^{m_{\nu+1}} \frac{1}{|\mu_k|} < (c_3 + 1) \cdot \sum_{k=0}^{\nu} \frac{1}{|\mu_{m_k}|},$$

where

$$m_0 := 1, \quad m_1 := m_1(1).$$

On the other hand, $\sum_{k=1}^{+\infty} \frac{1}{|\mu_k|} = +\infty$ by our assumption. This fact and (3.16) give us

$$(3.17) \quad \sum_{k=0}^{+\infty} \frac{1}{|\mu_{m_k}|} = +\infty.$$

Now by (3.12) and (3.17) we conclude that $\lim_{\nu \rightarrow +\infty} r_\nu = +\infty$ and the proof is complete. \blacksquare

4. CONSTRUCTION AND PROPERTIES OF THE DISKS

Fix the numbers $r_0, R_0, \theta_0, \theta_T, c_1, c_2, c_3, c_4$ which are defined in section 2 and subsections 3.1, 3.2, 3.3. For the rest of this section we fix a subsequence (μ_n) of (λ_n) satisfying the following:

$$1) |\mu_n|, |\mu_{n+1}| - |\mu_n| > c_1 \text{ for } n = 1, 2, \dots$$

$$2) \sum_{k=1}^{+\infty} \frac{1}{|\mu_k|} = +\infty.$$

Finally, on the basis of the above, we consider the partition \mathcal{P} constructed in subsection 3.3.

4.1. Construction of the disks.

Our goal in this subsection is to construct a certain family of pairwise disjoint disks, based on the previous partition \mathcal{P} of the sector S . This family points out how one can use Runge's theorem to conclude the Proposition 2.1. Let us describe, very briefly, the highlights of our argument. The main idea is to assign to each point w of the partition \mathcal{P} a suitable closed disk $B(w\mu(w), c_4)$ with center $w\mu(w)$ and radius c_4 (the radius will be the same for every member of the family of the disks), where $\mu(w)$ will be chosen from the sequence (μ_n) , so that on the one hand the disks $B(w\mu(w), c_4)$, $w \in \mathcal{P}$ are pairwise disjoint and on the other hand the union of the disks, $\cup_{w \in \mathcal{P}} B(w\mu(w), c_4)$ "almost fills" the sector S . It is evident that doing that, all the "good properties" of the partition established in the previous section will pass now to the family of the disks.

So, let us begin with the desired construction. We set

$$\mathcal{B} := \{z \in \mathbb{C} / |z| \leq c_4\}.$$

Let $w \in \mathcal{P}$ be a fixed point in \mathcal{P} . By the definition of \mathcal{P} there exist unique $r' \in \{r_0, r_1, \dots, r_{\nu_0}\}$, $m' \in \{1, m_1(1)+1, m_2+1, \dots, m_{\nu_0}+1\}$ such that $w \in \mathcal{P}_0^{r', m'}$. By definition, $\mathcal{P}_0^{r', m'} = \phi_{r'}(\Delta_{m'})$. So there exists unique $n \in \{0, 1, \dots, \nu_{m'}\}$ such that $w = r' e^{2\pi i \theta_n^{m'}}$. Now there exist unique $k \in \mathbb{N}$, $k \geq 1$ and $j \in \{0, 1, \dots, m_1(m') - m'\}$ such that $n = k(m_1(m') - m' + 1) + j$, so we define

$$\mu(w) := \mu_{m'+j}.$$

Thus we assign, in a unique way, a term of the sequence (μ_n) to each one from the points of \mathcal{P} . Finally we set

$$\mathcal{B}_w := \mathcal{B} + w\mu(w).$$

The desired family of disks is the following:

$$\mathfrak{D} := \{\mathcal{B}\} \cup \{\mathcal{B}_w : w \in \mathcal{P}\}.$$

4.2. Properties of the disks.

Lemma 4.1. *We have $\mathcal{B} \cap \mathcal{B}_w = \emptyset$ for every $w \in \mathcal{P}$.*

Proof. $c_3 = \frac{c_4}{r_0 c_2} > \frac{c_4}{r_0}$, since $c_2 \in (0, 1)$. So $2c_3 > 2\frac{c_4}{r_0}$ and in view of $c_1 = 4(c_3 + 1) > 2c_3$ we get

$$(4.1) \quad c_1 > \frac{2c_4}{r_0}.$$

Take $w \in \mathcal{P}$. The closed disks \mathcal{B} , \mathcal{B}_w are centered at, 0, $w\mu(w)$ respectively and they have the same radius c_4 . Hence, we have to show that $|w\mu(w)| > 2c_4$. Since $|w| \geq r_0$, it suffices to prove that $|\mu(w)| > \frac{2c_4}{r_0}$. Observe now that, by the definition of $\mu(w)$ in the previous subsection,

$$(4.2) \quad \mu(w) = \mu_n$$

for some positive integer $n \in \mathbb{N}$ and from the choice of (μ_n)

$$(4.3) \quad |\mu_n| > c_1 \text{ for every } n \in \mathbb{N}.$$

Now, (4.1), (4.2) and (4.3) imply $|\mu(w)| > 2\frac{c_4}{r_0}$ and this finishes the proof of the lemma. ■

Lemma 4.2. *Let $w_1, w_2 \in \mathcal{P}$ such that $|w_1| < |w_2|$. Then $\mathcal{B}_{w_1} \cap \mathcal{B}_{w_2} = \emptyset$.*

Proof. We have

$$m_0 = 1 < m_1(1) + 1,$$

$$m_2 = m_1(m_1(1) + 1) > m_1(1)$$

and generally

$$m_{\nu+1} = m_1(m_\nu + 1) > m_\nu \text{ for } \nu = 1, 2, \dots, \nu_0.$$

Since $w_1, w_2 \in \mathcal{P}$, we have $w_1 \in \mathcal{P}_0^{r_{\nu_1}, m_{\nu_1}+1}$, $w_2 \in \mathcal{P}_0^{r_{\nu_2}, m_{\nu_2}+1}$ for some $\nu_1, \nu_2 \in \{0, 1, \dots, \nu_0\}$ and so $|w_1| = r_{\nu_1}$, $|w_2| = r_{\nu_2}$. Our hypothesis $|w_1| < |w_2| \Leftrightarrow r_{\nu_1} < r_{\nu_2}$ and the fact that the sequence (r_ν) is strictly increasing gives us $\nu_1 < \nu_2$. Thus, $m_{\nu_1} + 1 < m_{\nu_2} + 1$, because the finite sequence (m_ν) , $\nu \in \{0, 1, \dots, \nu_0\}$ is strictly increasing; recall that $m_0 = 1$, $m_1 = m_1(1)$. By the definition of $\mu(w)$ for $w \in \mathcal{P}_0^{r', m'} \subset \mathcal{P}$ we get $\mu(w) = \mu_{m'+j}$ for some $j \in \{0, 1, \dots, m_1(m') - m'\}$, so

$|\mu_{m'}| \leq |\mu(w)| \leq |\mu_{m_1(m')}|$, since the sequence $(|\mu_n|)$ is strictly increasing. The fact that $w_1 \in \mathcal{P}_0^{r_{\nu_1}, m_{\nu_1}+1}$ implies

$$\begin{aligned} |\mu_{m_{\nu_1}+1}| &\leq |\mu(w_1)| \leq |\mu_{m_1(m_{\nu_1}+1)}| \\ &= |\mu_{m_{\nu_1}+1}| < |\mu_{m_{\nu_1+1}+1}| \leq |\mu_{m_{\nu_2}+1}|, \end{aligned}$$

since $\nu_1 + 1 \leq \nu_2$ and the sequence $(|\mu_n|)$ is strictly increasing (4.1). On the other hand we have $w_2 \in \mathcal{P}_0^{r_{\nu_2}, m_{\nu_2}+1}$, so

$$|\mu_{m_{\nu_2}+1}| \leq |\mu(w_2)| \leq |\mu_{m_{\nu_2}+1}|.$$

Hence, the last two inequalities above give

$$|\mu(w_1)| < |\mu(w_2)|,$$

which in turn implies

$$(4.4) \quad |w_2\mu(w_2)| > |w_1\mu(w_1)|.$$

By (4.4) and the hypothesis we get

$$\begin{aligned} |w_2\mu(w_2) - w_1\mu(w_1)| &\geq ||w_2\mu(w_2)| - |w_1\mu(w_1)|| \\ &= |w_2\mu(w_2)| - |w_1\mu(w_1)| > |w_1| |\mu(w_2)| - |w_1| |\mu(w_1)| \\ &\geq r_0(|\mu(w_2)| - |\mu(w_1)|) > r_0 c_1 > 2c_4, \end{aligned}$$

where the last inequality in the right hand side above follows from $c_1 > \frac{2c_4}{r_0}$, which is already established in Lemma 4.1. This shows that $\mathcal{B}_{w_1} \cap \mathcal{B}_{w_2} = \emptyset$. \blacksquare

Lemma 4.3. *Let $w_1, w_2 \in \mathcal{P}$ such that $w_1 \neq w_2$ and $|w_1| = |w_2|$. Then $\mathcal{B}_{w_1} \cap \mathcal{B}_{w_2} = \emptyset$.*

Proof. We distinguish two cases:

$$(i) \quad |\mu(w_1)| < |\mu(w_2)|.$$

In this case, by our hypothesis, we have

$$\begin{aligned} |w_2\mu(w_2) - w_1\mu(w_1)| &\geq ||w_2\mu(w_2)| - |w_1\mu(w_1)|| \\ &= |w_1| \cdot (|\mu(w_2)| - |\mu(w_1)|) \geq r_0 \cdot c_1 > 2c_4. \end{aligned}$$

Therefore $\mathcal{B}_{w_1} \cap \mathcal{B}_{w_2} = \emptyset$.

$$(ii) \quad |\mu(w_1)| = |\mu(w_2)|.$$

Since $w_1, w_2 \in \mathcal{P}$ it follows that $w_1 \in \mathcal{P}_0^{r_{\nu_1}, m_{\nu_1}+1}$, $w_2 \in \mathcal{P}_0^{r_{\nu_2}, m_{\nu_2}+1}$ for some $\nu_1, \nu_2 \in \{0, 1, \dots, \nu_0\}$. By the equalities $|w_1| = r_{\nu_1}$, $|w_2| = r_{\nu_2}$ and the hypothesis $|w_1| = |w_2|$ we conclude that $r_{\nu_1} = r_{\nu_2}$, which in turn implies $\nu_1 = \nu_2$, since the sequence (r_ν) is strictly increasing. Setting $\nu_1 = \nu_2 = \nu'$ we get $w_1, w_2 \in$

$\mathcal{P}_0^{r_{\nu'}, m_{\nu'}+1}$ for some $\nu' \in \{0, 1, \dots, \nu_0\}$, that is w_1, w_2 belong to the same partition of zero order. For simplicity we write $m_{\nu'} + 1 = m'$. We also set $r_{\nu'} = r'$. So, $w_1, w_2 \in \mathcal{P}_0^{r', m'}$ and the definition of the set $\mathcal{P}_0^{r', m'}$ gives us $w_1 = r' \cdot e^{2\pi i \theta_{n_1}^{(m')}}$, $w_2 = r' \cdot e^{2\pi i \theta_{n_2}^{(m')}}$ for some $n_1, n_2 \in \{0, 1, \dots, \nu_{m'}\}$, $n_1 \neq n_2$, since $w_1 \neq w_2$. Without loss of generality suppose that $n_1 < n_2$. Now, there exists a unique pair (k_1, j_1) , where $k_1 \in \mathbb{N}$, $j_1 \in \{0, 1, \dots, m_1(m') - m'\}$ and a unique pair (k_2, j_2) where $k_2 \in \mathbb{N}$ and $j_2 \in \{0, 1, \dots, m_1(m') - m'\}$ such that

$$(4.5) \quad n_1 = k_1(m_1(m') - m' + 1) + j_1$$

and

$$(4.6) \quad n_2 = k_2(m_1(m') - m' + 1) + j_2.$$

By definition, $\mu(w_1) = \mu_{m'+j_1}$ and $\mu(w_2) = \mu_{m'+j_2}$ and our hypothesis implies

$$|\mu(w_1)| = |\mu(w_2)| \Leftrightarrow \mu(w_1) = \mu(w_2).$$

So we have $j_1 = j_2 = j_0$ and

$$\begin{aligned} \theta_{n_1}^{(m')} &= \theta_{j_0}^{(m')} + k_1 \sigma_{m'}, \\ \theta_{n_2}^{(m')} &= \theta_{j_0}^{(m')} + k_2 \sigma_{m'}. \end{aligned}$$

Thus

$$(4.7) \quad \theta_{n_2}^{(m')} - \theta_{n_1}^{(m')} = (k_2 - k_1) \sigma_{m'}.$$

By (4.5), (4.6) and the fact that $n_1 < n_2$ and $j_1 = j_2$ we have $k_1 < k_2 \Rightarrow k_2 \geq k_1 + 1$. So, in view of (4.7) we arrive at

$$(4.8) \quad \theta_{n_2}^{(m')} - \theta_{n_1}^{(m')} \geq \sigma_{m'} > 0.$$

The previous imply the following bound.

$$\begin{aligned} |w_2 \mu(w_2) - w_1 \mu(w_1)| &= |\mu(w_1)| \cdot |w_1 - w_2| \geq \mu_{m'} \cdot |w_1 - w_2| \\ &= |\mu_{m'}| \cdot |r' \cdot e^{2\pi i \theta_{n_2}^{(m')}} - r' \cdot e^{2\pi i \theta_{n_1}^{(m')}}| \\ &= r' |\mu_{m'}| \cdot |e^{2\pi i \theta_{n_2}^{(m')}} - e^{2\pi i \theta_{n_1}^{(m')}}| \\ &= r' |\mu_{m'}| \cdot 2 \sin(\pi(\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')})) \\ (4.9) \quad &\geq r_0 \cdot |\mu_{m'}| \cdot 2 \sin(\pi(\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')})). \end{aligned}$$

Now, consider Jordan's inequality

$$\sin x > \frac{2}{\pi} x, \quad x \in \left(0, \frac{\pi}{2}\right).$$

We have

$$0 < \theta_{n_2}^{(m')} - \theta_{n_1}^{(m')} \leq \frac{1}{4} \Rightarrow 0 < \pi(\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')}) < \frac{\pi}{4}.$$

So, applying Jordan's inequality for

$$x = \pi(\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')})$$

we get

$$(4.10) \quad \sin(\pi(\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')})) > 2(\theta_{n_2}^{(m')} - \theta_{n_1}^{(m')}).$$

By (4.8), (4.9) and (4.10) it follows that

$$(4.11) \quad |w_2\mu(w_2) - w_1\mu(w_1)| > 4r_0|\mu_{m'}| \cdot \sigma_{m'}.$$

The definition of the number $\sigma_{m'}$ and relation (3.6) of Lemma 3.1 yield

$$\sigma_{m'} = c_2 \cdot \sum_{k=m'}^{m_1(m')} \frac{1}{|\mu_k|}.$$

By this fact, inequality (4.11) and the definition of the number $m_1(m')$ we get

$$(4.12) \quad \begin{aligned} |w_2\mu(w_2) - w_1\mu(w_1)| &> 4r_0|\mu_{m'}| \cdot c_2 \sum_{k=m'}^{m_1(m')} \frac{1}{|\mu_k|} \\ &> 4r_0|\mu_{m'}| \cdot c_2 \frac{c_3}{|\mu_{m'}|} = 4r_0c_2c_3. \end{aligned}$$

Recall that $c_3 = \frac{c_4}{r_0c_2}$. So

$$4r_0c_2c_3 = 4r_0c_2 \cdot \frac{c_4}{r_0c_2} = 4c_4 > 2c_4.$$

The last bound along with (4.12) give $\mathcal{B}_{w_1} \cap \mathcal{B}_{w_2} = \emptyset$ and the proof of the lemma is complete. \blacksquare

By Lemmas 4.1, 4.2, 4.3 we conclude the following

Corollary 4.4. *The family $\mathcal{D} := \{\mathcal{B}\} \cup \{\mathcal{B}_w : w \in \mathcal{P}\}$ consists of pairwise disjoint disks.*

5. PROOF OF LEMMA 2.3

Let $j_1, s_1, k_1 \in \mathbb{N}$ be fixed. Our aim is to prove that the set $\bigcup_{m=1}^{+\infty} E(m, j_1, s_1, k_1)$ is dense in $H(\mathbb{C})$. For simplicity we write $p_{j_1} = p$. Fix $g \in H(\mathbb{C})$, a compact set $C \subseteq \mathbb{C}$ and $\varepsilon_0 > 0$. We seek $f \in H(\mathbb{C})$ and a positive integer m_1 such that

$$(5.1) \quad f \in E(m_1, j_1, s_1, k_1)$$

and

$$(5.2) \quad \sup_{z \in C} |f(z) - g(z)| < \varepsilon_0.$$

Fix $R_1 > 0$ sufficiently large so that

$$C \cup \{z \in \mathbb{C} \mid |z| \leq k_1\} \subset \{z \in \mathbb{C} \mid |z| \leq R_1\}$$

and then choose $0 < \delta_0 < 1$ such that

$$(5.3) \quad \text{if } |z| \leq R_1 \text{ and } |z - w| < \delta_0, \quad w \in \mathbb{C}, \text{ then } |p(z) - p(w)| < \frac{1}{2s_1}.$$

Define

$$\begin{aligned} \mathcal{B} &:= \{z \in \mathbb{C} \mid |z| \leq R_1 + \delta_0\}, \\ c_4 &:= R_1 + \delta_0, \quad c_2 := \frac{\delta_0}{2(2R_0\pi + 1)}, \\ c_3 &= \frac{c_4}{r_0 c_2} = \frac{R_1 + \delta_0}{r_0 \frac{\delta_0}{2(2R_0\pi + 1)}} = \frac{2(R_1 + \delta_0)(2R_0\pi + 1)}{r_0 \delta_0}, \\ c_1 &= 4(c_3 + 1) = 4 \cdot \left(\frac{2(R_1 + \delta_0)(2R_0\pi + 1)}{r_0 \delta_0} + 1 \right). \end{aligned}$$

After the definition of the above numbers we choose a subsequence (μ_n) of (λ_n) such that

$$(i) \quad |\mu_n| > c_1, \quad |\mu_{n+1}| - |\mu_n| > c_1 \text{ for } n = 1, 2, \dots \text{ and}$$

$$(ii) \quad \sum_{n=1}^{+\infty} \frac{1}{|\mu_n|} = +\infty.$$

On the basis of the fixed numbers $r_0, R_0, \theta_0, \theta_T, c_1, c_2, c_3, c_4$ and the choice of the sequence (μ_n) we define the set L as follows:

$$L := \mathcal{B} \cup \left(\bigcup_{w \in \mathcal{P}} \mathcal{B}_w \right),$$

where the partition \mathcal{P} and the discs \mathcal{B}_w , $w \in \mathcal{P}$ are constructed in Sections 3 and 4 respectively. By Corollary 4.4, the family \mathfrak{D} consists of pairwise disjoint disks. Therefore the compact set L has connected complement. This property is needed in order to apply Mergelyan's theorem. We now define the function h on the compact set L , $h : L \rightarrow \mathbb{C}$ by

$$h(z) = \begin{cases} g(z), & z \in \mathcal{B} \\ p(z - w\lambda(w)), & z \in \mathcal{B}_w, w \in \mathcal{P}. \end{cases}$$

By Mergelyan's theorem [43] there exists an entire function f (in fact a polynomial) such that

$$(5.4) \quad \sup_{z \in L} |f(z) - h(z)| < \min \left\{ \frac{1}{2s_1}, \varepsilon_0 \right\}.$$

The definition of h and (5.4) give

$$\sup_{z \in C} |f(z) - g(z)| \leq \sup_{z \in \mathcal{B}} |f(z) - g(z)| = \sup_{z \in L} |f(z) - h(z)| < \varepsilon_0,$$

which implies the desired inequality (5.2). It remains to show (5.1).

Let $a \in S$. Then $a = re^{2\pi i\theta}$ for some $r \in [r_0, R_0]$ and $\theta \in [\theta_0, \theta_T]$. There exists a unique $n_0 \in \{0, 1, \dots, \nu_0 - 1\}$ such that either $r_{n_0} \leq r < r_{n_0+1}$ or $r_{\nu_0} \leq r \leq R_0$. We set

$$r_1 := r_{n_0}, \quad r_2 := r_{n_0+1} \quad \text{if } r_{n_0} \leq r < r_{n_0+1}$$

and

$$r_1 := r_{\nu_0}, \quad r_2 := R_0 \quad \text{if } r_{\nu_0} \leq r \leq R_0.$$

By the construction of the partition \mathcal{P} there exists a unique $m' \in \mathbb{N}$ such that $\mathcal{P}_0^{r_1, m'} \subset \mathcal{P}$. In addition, there exists unique $\rho \in \{0, 1, \dots, \nu_{m'} - 1\}$ such that

$$\text{either } \theta_\rho^{(m')} \leq \theta < \theta_{\rho+1}^{(m')} \quad \text{or} \quad \theta_{\nu_{m'}}^{(m')} \leq \theta \leq \theta_T.$$

Define now

$$\theta_1 := \theta_\rho^{(m')}, \theta_2 := \theta_{\rho+1}^{(m')} \quad \text{if } \theta_\rho^{(m')} \leq \theta < \theta_{\rho+1}^{(m')}$$

and

$$\theta_1 := \theta_{\nu_{m'}}^{(m')}, \theta_2 := \theta_T \quad \text{if } \theta_{\nu_{m'}}^{(m')} \leq \theta \leq \theta_T$$

and then set

$$w_0 := r_1 \cdot e^{2\pi i\theta_1} \in \mathcal{P}_0^{r_1, m'}.$$

We shall prove now that for every $z \in \mathbb{C}$ with $|z| \leq R_1$, $z + a\mu(w_0) \in \mathcal{B}_{w_0}$. Recall that $\mathcal{B}_{w_0} := \mathcal{B} + w_0\mu(w_0) = \overline{D}(w_0\mu(w_0), R_1 + \delta_0)$. It suffices to prove that

$$(5.5) \quad |(z + a\mu(w_0)) - w_0\mu(w_0)| < R_1 + \delta_0 \quad \text{for } |z| \leq R_1.$$

For $|z| \leq R_1$ we have,

$$(5.6) \quad \begin{aligned} |(z + a\mu(w_0)) - w_0\mu(w_0)| &\leq R_1 + |\mu(w_0)| |a - w_0| \\ &= R_1 + |\mu(w_0)| \cdot |r \cdot e^{2\pi i\theta} - r_1 e^{2\pi i\theta_1}|. \end{aligned}$$

By (5.5) and (5.6) it suffices to prove

$$(5.7) \quad |\mu(w_0)| \cdot |re^{2\pi i\theta} - r_1 e^{2\pi i\theta_1}| < \delta_0.$$

We now have

$$\begin{aligned}
|re^{2\pi i\theta} - r_1e^{2\pi i\theta_1}| &= |re^{2\pi i\theta} - r_1e^{2\pi i\theta} + r_1e^{2\pi i\theta} - r_1e^{2\pi i\theta_1}| \\
&\leq |re^{2\pi i\theta} - r_1e^{2\pi i\theta}| + |r_1e^{2\pi i\theta} - r_1e^{2\pi i\theta_1}| \\
&\leq |r - r_1| + r_1|e^{2\pi i\theta} - e^{2\pi i\theta_1}| \\
&\leq |r_1 - r_2| + R_0 \cdot 2\sin(\pi(\theta_1 - \theta)) \\
&\leq (r_2 - r_1) + R_0 2\sin(\pi(\theta_2 - \theta_1)) \\
&< (r_2 - r_1) + 2R_0\pi(\theta_2 - \theta_1) \\
&\leq 2R_0\pi \cdot \frac{c_2}{|\mu(w_0)|} + \frac{c_2}{|\mu(w_0)|} \\
&= (2R_0\pi + 1)c_2 \frac{1}{|\mu(w_0)|} \\
&= (2R_0\pi + 1) \cdot \frac{\delta_0}{2(2R_0\pi + 1)} \cdot \frac{1}{|\mu(w_0)|} = \frac{\delta_0}{2|\mu(w_0)|}
\end{aligned}$$

which implies (5.7). So we proved that for every $z \in \mathbb{C}$, $|z| \leq R_1$

$$(5.8) \quad z + a\mu(w_0) \in \mathcal{B}_{w_0}.$$

By the definition of h and (5.8) we have that for every $z \in \mathbb{C}$ with $|z| \leq R_1$

$$(5.9) \quad |f(z + a\mu(w_0)) - p(z + \mu(w_0)(re^{2\pi i\theta} - r_1e^{2\pi i\theta_1}))| < \frac{1}{2s_1}.$$

Take any $z \in \mathbb{C}$ with $|z| \leq R_1$. By (5.3) and (5.7)

$$(5.10) \quad |p(z + \mu(w_0)(re^{2\pi i\theta} - r_1e^{2\pi i\theta_1})) - p(z)| < \frac{1}{2s_1}$$

and the triangle inequality gives

$$\begin{aligned}
|f(z + a\mu(w_0)) - p(z)| &\leq |f(z + a\mu(w_0)) - p(z + \mu(w_0)(re^{2\pi i\theta} - r_1e^{2\pi i\theta_1}))| \\
(5.11) \quad &+ |p(z + \mu(w_0)(re^{2\pi i\theta} - r_1e^{2\pi i\theta_1})) - p(z)|.
\end{aligned}$$

Using (5.9), (5.10), (5.11) we arrive at

$$|f(z + a\mu(w_0)) - p(z)| < \frac{1}{s_1}$$

and since $k_1 \leq R_1$ it readily follows that

$$(5.12) \quad \sup_{|z| \leq k_1} |f(z + a\mu(w_0)) - p(z)| < \frac{1}{s_1}.$$

Set

$$m_1 := \max\{n \in \mathbb{N} | \lambda_n = \mu(w), \text{ for some } w \in \mathcal{P}\}$$

and observe that the definition of m_1 is independent from $a \in S$. Thus, by the previous we conclude that for every $a \in S$ there exists some $n \in \mathbb{N}$, $n \leq m_1$ such that

$$\sup_{|z| \leq k_1} |f(z + a\lambda_n) - p(z)| < \frac{1}{s_1},$$

where $f \in H(\mathbb{C})$, since f is a polynomial. This completes the proof of the lemma. \blacksquare

6. EXAMPLES OF SEQUENCES $\Lambda := (\lambda_n)$ SATISFYING CONDITION Σ

By the remark in [28] we have a sample of first examples satisfying condition (Σ) :

$$\lambda_n = n, \quad \lambda_n = n(\log n)^p \text{ for } p \leq 1, \quad \lambda_n = n \log n \log \log n.$$

In all the above examples we also have $\left| \frac{\lambda_{n+1}}{\lambda_n} \right| \rightarrow 1$ as $n \rightarrow +\infty$. However, for sequences (λ_n) , such that $\lambda_n \rightarrow \infty$ and $\left| \frac{\lambda_{n+1}}{\lambda_n} \right| \rightarrow 1$ we have that the conclusion of Theorem 1.1 holds by the main result in [48]. It is our aim to show that there exist sequences (λ_n) , such that: $\lambda_n \rightarrow \infty$, (λ_n) satisfies condition (Σ) and the ratio $\left| \frac{\lambda_{n+1}}{\lambda_n} \right|$ does not tend to 1.

Let us see things more specifically. Consider a sequence $\Lambda = (\lambda_n)$ of non-zero complex numbers and define

$$\mathcal{B}(\Lambda) := \left\{ a \in [0, +\infty] \mid \exists (\mu_n) \subset \Lambda \text{ with } a = \limsup_n \left| \frac{\mu_{n+1}}{\mu_n} \right| \right\},$$

$$i(\Lambda) := \inf \mathcal{B}(\Lambda).$$

Clearly

$$i(\Lambda) \in [0, +\infty]$$

and

$$\text{if } \lambda_n \rightarrow \infty \text{ then } \mathcal{B}(\Lambda) \subset [1, +\infty] \text{ and } i(\Lambda) \in [1, +\infty].$$

We say that a sequence of non-zero complex numbers $\Lambda = (\lambda_n)$ satisfies condition (Σ') if $i(\Lambda) = 1$. In [48] we established the following result.

If $\Lambda = (\lambda_n)$ is a sequence of non-zero complex numbers such that $\lambda_n \rightarrow \infty$ and Λ satisfies condition (Σ') , then the conclusion of Theorem 1.1 holds.

In view of the above result the following question arises naturally.

Question 6.1. *Let $\Lambda = (\lambda_n)$ be a sequence of non-zero complex numbers such that $\lambda_n \rightarrow \infty$ and $i(\Lambda) > 1$. Does the conclusion of Theorem 1.1 hold?*

It is quite surprising, at least to us, that the answer to the above question is sometimes yes and sometimes no!. In what follows we shall exhibit examples of sequences admitting a positive answer to this question. Results going to the opposite direction are established in [32]. In particular, the main result in [32] is the following: if $\Lambda = (\lambda_n)$ is a sequence of non-zero complex numbers such that $\liminf_n \frac{|\lambda_{n+1}|}{|\lambda_n|} > 2$ (hence $i(\Lambda) > 2$) then $\bigcap_{a \in \mathbb{C} \setminus \{0\}} HC(\{T_{\lambda_n a}\}) = \emptyset$.

Below we construct specific examples of sequences $\Lambda = (\lambda_n)$ such that $\lambda_n \rightarrow \infty$, $i(\Lambda) = M$ for any fixed positive number $M > 1$ and Λ satisfies (Σ) . By this result we complete our goals in this paper that are the following three.

- Firstly, we give affirmative reply to Question 1 of [28].
- Secondly, for certain sequences, we also give a positive answer to Question 6.1.
- Thirdly, we exhibit a variety of examples of sequences $\Lambda = (\lambda_n)$ of non-zero complex numbers with $\lambda_n \rightarrow \infty$ such that Λ satisfies condition (Σ) and it does not satisfy condition (Σ') .

The above discussion shows that the problem of deciding whether a sequence $\Lambda = (\lambda_n)$, such that $\lambda_n \rightarrow \infty$ and $i(\Lambda) = M$ for some $M > 1$ satisfies the conclusion of Theorem 1.1 is quite delicate and needs further study.

Proposition 6.1. *For every $M > 1$ there exists a sequence $\Lambda = (\lambda_n)$ such that $\lambda_n \rightarrow \infty$, $i(\Lambda) = M$ and condition (Σ) holds for Λ . Thus, for every $M > 1$ there exists a sequence of non-zero complex numbers $\Lambda = (\lambda_n)$ such that $\lambda_n \rightarrow \infty$ as $n \rightarrow +\infty$, $i(\Lambda) = M$ and $\bigcap_{a \in \mathbb{C} \setminus \{0\}} HC(\{T_{\lambda_n a}\})$ is a G_δ and dense subset of $H(\mathbb{C})$.*

Proof. Fix a positive number $M_0 > 1$. We shall construct a sequence of non-zero complex numbers $\Lambda = (\lambda_n)$ such that $\lambda_n \rightarrow \infty$, $i(\Lambda) = M_0$ and condition (Σ) holds for Λ . The sequence Λ will be a strictly increasing sequence of positive numbers such that $\lambda_n \rightarrow +\infty$ as $n \rightarrow +\infty$.

We construct inductively a countable family $\{\mathfrak{D}_n\}$, $n = 1, 2, \dots$ of sets $\mathfrak{D}_n \subset [1, +\infty)$ according to the following rules.

- (i) $\mathfrak{D}_1 = \{1\}$.
- (ii) $\mathfrak{D}_n = \{a_n + \nu \mid \nu = 0, 1, \dots, ([a_n] + 1)!\}$, $n = 1, 2, \dots$
- (iii) $\min \mathfrak{D}_{n+1} = M_0 \cdot \max \mathfrak{D}_n$ for each $n = 1, 2, \dots$,

where $a_n = \min \mathfrak{D}_n$ and $[x]$ denotes the integer part of the real number x as usual. Observe that every $n, m \in \mathbb{N}$, $n \neq m$, $\mathfrak{D}_n \cap \mathfrak{D}_m = \emptyset$. Set

$$\tilde{\Lambda} = \bigcup_{n=1}^{+\infty} \mathfrak{D}_n.$$

We define the sequence $\Lambda = (\lambda_n)$ to be the enumeration of $\tilde{\Lambda}$ by the natural order.

It is obvious that $\lambda_n \neq 0 \ \forall \ n \in \mathbb{N}$, $\lim_{n \rightarrow +\infty} \lambda_n = +\infty$, and (λ_n) is a strictly increasing sequence of positive numbers. We prove now the following

Claim 1: For every subsequence $\mu = (\mu_n)$ of Λ we have $\limsup_{n \rightarrow +\infty} \frac{\mu_{n+1}}{\mu_n} \geq M_0$.

Proof. Firstly we prove that for every natural number $m \in \mathbb{N}$, there exists some $N \in \mathbb{N}$, $N \geq m$ such that

$$\frac{\mu_{N+1}}{\mu_N} \geq M_0.$$

So, take any $m \in \mathbb{N}$ and let m_1 be the unique natural number such that $\mu_{m_1} \in \mathfrak{D}_{m_1}$. Setting $A_{m_1} := \{n \in \mathbb{N} | \mu_n \in \mathfrak{D}_{m_1}\}$, it is obvious that $A_{m_1} \neq \emptyset$, since $m \in A_{m_1}$. We set $m_2 := \max A_{m_1}$. Then $\mu_{m_2+1} \notin \mathfrak{D}_{m_1}$ and so $\mu_{m_2+1} \geq \min \mathfrak{D}_{m_1+1}$. We have $\mu_{m_2} \leq \max \mathfrak{D}_{m_1}$, thus

$$\frac{\mu_{m_2+1}}{\mu_{m_2}} \geq \frac{\min \mathfrak{D}_{m_1+1}}{\max \mathfrak{D}_{m_1}} = M_0 \quad \text{and} \quad m_2 \geq m_1.$$

So we proved that for every $m \in \mathbb{N}$, there exists some $N \geq m$ such that $\frac{\mu_{N+1}}{\mu_N} \geq M_0$. We incorporate the last fact into an inductive argument and obtain the following. For $m = 1$ there exists $k_1 \in \mathbb{N}$, $k_1 \geq 1$ such that $\frac{\mu_{k_1+1}}{\mu_{k_1}} \geq M_0$. For $m = k_1 + 1$, there exists some $k_2 \geq k_1 + 1$ (especially $k_2 > k_1$) such that $\frac{\mu_{k_2+1}}{\mu_{k_2}} \geq M_0$. Suppose that for some $\nu \in \mathbb{N}$ we have found some $k_\nu \in \mathbb{N}$ such that $\frac{\mu_{k_\nu+1}}{\mu_{k_\nu}} \geq M_0$. Then for $m = k_\nu + 1$ there exists some $k_{\nu+1} \geq k_\nu + 1$ (especially $k_{\nu+1} > k_\nu$) such that $\frac{\mu_{k_{\nu+1}+1}}{\mu_{k_{\nu+1}}} \geq M_0$. Therefore we obtain a subsequence (μ_{k_ν}) , $\nu = 1, 2, \dots$ of (μ_n) such that $k_{\nu+1} > k_\nu$ for each $\nu = 1, 2, \dots$ and $\frac{\mu_{k_\nu+1}}{\mu_{k_\nu}} \geq M_0$. This gives $\limsup_{\nu \rightarrow +\infty} \frac{\mu_{k_\nu+1}}{\mu_{k_\nu}} \geq M_0$, which in turn implies

$$\limsup_{n \rightarrow +\infty} \frac{\mu_{n+1}}{\mu_n} \geq M_0.$$

This completes the proof of Claim 1.

Claim 2: $\limsup_{n \rightarrow +\infty} \frac{\lambda_{n+1}}{\lambda_n} = M_0$.

Proof. Let $n \in \mathbb{N}$. If $\lambda_n, \lambda_{n+1} \in \mathfrak{D}_m$ for some positive integer m , then by the construction of \mathfrak{D}_m we have

$$(6.1) \quad \lambda_{n+1} = \lambda_n + 1 \Rightarrow \frac{\lambda_{n+1}}{\lambda_n} = 1 + \frac{1}{\lambda_n}.$$

If there is no $m \in \mathbb{N}$ such that $\lambda_n, \lambda_{n+1} \in \mathfrak{D}_m$, then this can happen only if $\lambda_n = \max \mathfrak{D}_m$ and $\lambda_{n+1} = \min \mathfrak{D}_{m+1}$ for some $m \in \mathbb{N}$, hence

$$(6.2) \quad \frac{\lambda_{n+1}}{\lambda_n} = M_0.$$

By (6.1), (6.2) and since $\lambda_n \rightarrow +\infty$ the conclusion follows. This completes the proof of Claim 2.

Claims 1 and 2 imply that $i(\Lambda) = M_0$.

Claim 3: The sequence Λ satisfies condition (Σ) .

Proof. Fix some natural number $N_0 \geq 2$. We will show that there exists a subsequence (μ_n) of Λ such that

(i) $\mu_{n+1} - \mu_n > N_0$ for every $n = 1, 2, \dots$ and

(ii) $\sum_{n=1}^{+\infty} \frac{1}{\mu_n} = +\infty$.

Recall that $a_n = \min \mathfrak{D}_n > 1$ for every $n \geq 2$. Since

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n \right) = \gamma,$$

where $\gamma \simeq 0,57722156649\dots$ is the Euler constant, there exists some natural number $n_0 \in \mathbb{N}$ such that

$$-\frac{1}{2} < \sum_{k=1}^n \frac{1}{k} - \log n - \gamma < \frac{1}{2} \quad \text{for } n \geq n_0 > 2.$$

Let some $m, n \in \mathbb{N}$, $m > n \geq n_0$. Then we have

$$\begin{aligned} \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{m} &= \sum_{k=n+1}^m \frac{1}{k} = \sum_{k=1}^m \frac{1}{k} - \sum_{k=1}^n \frac{1}{k} \\ &= \left(\sum_{k=1}^m \frac{1}{k} - \log m - \gamma \right) - \left(\sum_{k=1}^n \frac{1}{k} - \log n - \gamma \right) \\ &\quad + \log m - \log n > \log \frac{m}{n} - 1 \\ &= \log \frac{m}{n} + \log e^{-1} = \log \left(\frac{m}{n} \cdot e^{-1} \right) \\ (6.3) \quad &= \log \left(\frac{m}{ne} \right). \end{aligned}$$

It is easy to show that $a_n > n$ for $n \geq 2$. Set $n_1 := \max\{n_0, N_0\} + 2$. Let now some $n \in \mathbb{N}$ with $n \geq n_1$. Recall that

$$\begin{aligned}\mathfrak{D}_n &= \{a_n, a_n + 1, \dots, a_n + ([a_n] + 1)!\} \\ &= \{a_n + j | j = 0, 1, \dots, ([a_n] + 1)!\}\end{aligned}$$

Setting $N_1 := N_0 + 1$ we obtain

$$\begin{aligned}(6.4) \quad & \frac{1}{a_n} + \frac{1}{a_n + N_1} + \frac{1}{a_n + 2N_1} + \dots + \frac{1}{a_n + \frac{([a_n] + 1)!}{N_1} \cdot N_1} \\ & > \frac{1}{N_1 a_n} + \frac{1}{N_1 a_n + N_1} + \frac{1}{N_1 a_n + 2N_1} + \dots + \frac{1}{N_1 a_n + N_1 \cdot \frac{([a_n] + 1)!}{N_1}} \\ & = \frac{1}{N_1} \cdot \sum_{j=0}^{\frac{([a_n] + 1)!}{N_1}} \frac{1}{a_n + j} > \frac{1}{N_1} \cdot \sum_{j=0}^{\frac{([a_n] + 1)!}{N_1}} \frac{1}{([a_n] + 1) + j}.\end{aligned}$$

We write for simplicity $\nu = [a_n] + 1$. So by (6.3), (6.4) we get

$$(6.5) \quad \sum_{k=0}^{\frac{\nu!}{N_1}} \frac{1}{a_n + kN_1} > \frac{1}{N_1} \cdot \log \left(\frac{\nu + \frac{\nu!}{N_1}}{(\nu - 1)e} \right) > \frac{1}{N_1} \cdot \log \left(\frac{(\nu - 1)!}{N_1 e} \right).$$

We will show that

$$\frac{1}{N_1} \cdot \log \left(\frac{(\nu - 1)!}{N_1 e} \right) > \nu$$

for ν big enough. It follows that

$$\begin{aligned}\frac{1}{N_1} \cdot \log \left(\frac{(\nu - 1)!}{N_1 e} \right) > \nu &\Leftrightarrow \log \left(\frac{(\nu - 1)!}{N_1 e} \right) > N_1 \nu \\ &\Leftrightarrow (\nu - 1)! > N_1 e \cdot e^{N_1 \nu} = N_1 \cdot e^{N_1 \nu + 1}.\end{aligned}$$

Let us consider the sequence $\gamma_\nu = \frac{(\nu - 1)!}{N_1 e^{N_1 \nu + 1}}$. By the ratio criterion for (γ_ν) we have

$$\frac{\gamma_{\nu+1}}{\gamma_\nu} = \frac{\frac{\nu!}{N_1 e^{N_1(\nu+1)+1}}}{\frac{(\nu - 1)!}{N_1 e^{N_1 \nu + 1}}} = \frac{\nu! \cdot e^{N_1 \nu + 1}}{(\nu - 1)! \cdot e^{N_1(\nu+1)+1}} = \frac{\nu}{e^{N_1}}.$$

So $\lim_{\nu \rightarrow +\infty} \left(\frac{\gamma_{\nu+1}}{\gamma_\nu} \right) = +\infty$ which implies that there exists some $n_2 \geq n_1$ such that $\gamma_n > 1$ for $n \geq n_2$ or equivalently

$$(6.6) \quad \frac{1}{N_1} \cdot \log \left(\frac{(n - 1)!}{N_1 e} \right) > n, \quad n \geq n_2.$$

Thus by (6.5) and (6.6) we have:

$$\sum_{k=0}^{\frac{\nu!}{N_1}} \frac{1}{a_n + kN_1} > [a_n] + 1 \quad \text{for } n \geq n_2.$$

Now for $n \geq n_2$ define the set

$$\mathfrak{D}'_n := \left\{ a_n, a_n + N_1, a_n + 2N_1, \dots, a_n + \frac{([a_n] + 1)!}{N_1} \cdot N_1 \right\},$$

and consider the union

$$\mathfrak{D}' := \bigcup_{n \geq n_2} \mathfrak{D}'_n.$$

Let (μ_n) be the sequence we get when we enumerate \mathfrak{D}' by its natural order. Clearly (μ_n) is a subsequence of Λ and satisfies the desired properties (i) and (ii). This completes the proof of Claim 3 and hence that of Proposition 6.1 using Theorem 1.1. ■

Corollary 6.1. *There exists a sequence $\Lambda = (\lambda_n)$ of non-zero complex numbers with $\lambda_n \rightarrow \infty$ such that Λ satisfies condition (Σ) and it does not satisfy condition (Σ') .*

Proof. Every sequence $\Lambda = (\lambda_n)$ of non-zero complex numbers with $\lambda_n \rightarrow \infty$ which satisfies the conclusion of Proposition 6.1, clearly does not satisfy (Σ') . ■

We point out that sequences of the form (n^2) , (n^3) , $(n^4) \dots$, satisfy condition (Σ') but they do not satisfy (Σ) . To complete the picture we observe that there are sequences with sufficiently slow growth, such as (n) , (\sqrt{n}) , $(\log(n+1))$, $(\log \log(n+1))$, that satisfy both conditions (Σ) and (Σ') . Hence, neither (Σ) nor (Σ') implies the other and, in addition, they have non-empty intersection. This in turn shows that Theorem 1.1 does not follow by the main result in [48] and vice versa.

We close the paper with a question which kindly posed to us by the referee.

Question 6.2. *If (λ_n) is a sequence of non-zero complex numbers, $\lambda_n \rightarrow \infty$ such that $\bigcap_{a \in \mathbb{C} \setminus \{0\}} HC(\{T_{\lambda_n a}\}) \neq \emptyset$ what can be said about the growth of common hypercyclic entire functions for the collection of sequences $(T_{\lambda_n a})$, $a \in \mathbb{C} \setminus \{0\}$? To answer such a question, we should specify the sequence (λ_n) . For instance, what happens when $\lambda_n = n^2$, $\lambda_n = n \log(n+1)$, $\lambda_n = n^3$, etc.?*

Results concerning permissible and optimal growth rates for hypercyclic entire functions with respect to the translation operator T_a , $a \in \mathbb{C} \setminus \{0\}$, as well as similar results for differential operators acting on various function spaces can be found in [2], [3], [5], [15], [17], [18], [20], [26], [33], [35], [36], [37], [39], [45].

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